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ON OPTIMAL STABILIZATION OF MOTION WITH RESPECT TO A PART OF VARIABLES

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The criterions of optimal stabilization of motion with respect to a part of the variables which are established here, modify the theorems of Krasovskii [1] and Rumiantsev [2,3]. Application of these criteria to autonomous systems is studied and an example given.

1. Let us consider a system of differential equations of perturbed motion of a controlled system

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}, \mathbf{u}) \quad (\mathbf{X}(t, \mathbf{0}, \mathbf{0}) \equiv \mathbf{0}) \tag{1.1}$$
$$\mathbf{x} = (y_1, \ldots, y_m, z_1, \ldots, z_p), \quad \mathbf{u} = (u_1, \ldots, u_r),$$
$$m > 0, \ p \geqslant 0, \ n = m + p, \ r > 0$$

We choose a certain class $K = \{\mathbf{u}(t, \mathbf{x})\}$ of controls $\mathbf{u}(t, \mathbf{x})$ continuous in the region

$$t \geqslant 0, \quad \| \mathbf{y} \| \leqslant H > 0, \quad 0 \leqslant \| \mathbf{z} \| < \infty \tag{1.2}$$

and assume that for any $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) \in K$

a) the right hand sides of the system (1.1) are continuous in the region (1.2) and satisfy the conditions of uniqueness of the solution;

b) solutions of the system (1.1) are $\mathbf{z}$-continuable, i.e. every solution $\mathbf{x}(t)$ is defined for all $t \geqslant 0$ for which $\| \mathbf{y}(t) \| \leqslant H$.

We use, as the control quality criterion, the condition of minimum of the integral [1]

$$J = \int_{t_0}^{\infty} \omega(t, \mathbf{x}[t], \mathbf{u}[t])\, dt, \quad \omega \geqslant 0 \tag{1.3}$$

for all $\mathbf{u}(t, \mathbf{x}) \in K$. The problem of optimal $\mathbf{y}$-stabilization [2,4] in class K consists of finding a function $\mathbf{u} = \mathbf{u}^\circ(t, \mathbf{x}) \in K$ ensuring the asymptotic $\mathbf{y}$-stability of the motion $\mathbf{x} = 0$, and the following inequality must hold for any function $\mathbf{u} = \mathbf{u}^*(t, \mathbf{x}) \in K$ satisfying this condition:

$$\int_{t_0}^{\infty} \omega(t, \mathbf{x}^\circ[t], \mathbf{u}^\circ[t])\, dt \leqslant \int_{t_0}^{\infty} \omega(t, \mathbf{x}^*[t], \mathbf{u}^*[t])\, dt$$

for $t_0 \geqslant 0$, $\mathbf{x}^\circ[t_0] = \mathbf{x}^*[t_0] = \mathbf{x}_0$, $\| \mathbf{x}_0 \| \leqslant \lambda = \text{const}$.

2. Following [1] we adopt the notation

$$B[V, t, \mathbf{x}, \mathbf{u}] = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}} \cdot \mathbf{X}(t, \mathbf{x}, \mathbf{u}) + \omega(t, \mathbf{x}, \mathbf{u}) \tag{2.1}$$

Theorem 1. Assume that the functions $\mathbf{u} = \mathbf{u}^\circ(t, \mathbf{x}) \in K$ and a function

$V(t, \mathbf{x})$ exist and satisfy the following conditions:

1) when $\mathbf{u} = \mathbf{u}^\circ(t, \mathbf{x})$, the motion $\mathbf{x} = 0$ is asymptotically $\mathbf{y}$-stable;

2) $B[V, t, \mathbf{x}, \mathbf{u}^\circ(t, \mathbf{x})] = 0$;

3) $B[V, t, \mathbf{x}, \mathbf{u}(t, \mathbf{x})] \geqslant 0$ for any $\mathbf{u}(t, \mathbf{x}) \in K$;

4) the following inequality holds for every control $\mathbf{u}^*(t, x) \in K$ ensuring the asymptotic $\mathbf{y}$-stability of the motion $\mathbf{x} = 0$:

$$\lim_{t\to\infty} V(t, \mathbf{x}^\circ[t]) \geqslant \lim_{t\to\infty} V(t, \mathbf{x}^*[t]) \tag{2.2}$$

(where we assume that the limits appearing in (2.2) exist).

Then the function $\mathbf{u} = \mathbf{u}^\circ(t, \mathbf{x})$ solves the problem of optimal $\mathbf{y}$-stabilization in class K.

Proof. By virtue of condition 2) of the theorem the relation $dV(t, \mathbf{x}^\circ[t]) / dt = -\omega(t, \mathbf{x}^\circ[t], \mathbf{u}^\circ[t])$ holds. Integrating this relation we obtain

$$V(t_0, \mathbf{x}_0) = \int_{t_0}^{\infty} \omega(t, \mathbf{x}^\circ[t], \mathbf{u}^\circ[t])\, dt + \lim_{t\to\infty} V(t, \mathbf{x}^\circ[t]) \tag{2.3}$$

By virtue of condition 3) of the theorem the inequality $dV(t, \mathbf{x}^*[t]) / dt \geqslant -\omega(t, \mathbf{x}^*[t])$ holds for the function $\mathbf{u}^*(t, \mathbf{x}) \in K$ satisfying the condition 4). Integrating this inequality we obtain

$$V(t_0, \mathbf{x}_0) \leqslant \int_{t_0}^{\infty} \omega(t, \mathbf{x}^*[t], \mathbf{u}^*[t])\, dt + \lim_{t\to\infty} V(t, \mathbf{x}^*[t]) \tag{2.4}$$

From (2.3) and (2.4) we have, by virtue of (2.2),

$$\int_{t_0}^{\infty} \omega(t, \mathbf{x}^\circ[t], \mathbf{u}^\circ[t])\, dt \leqslant \int_{t_0}^{\infty} \omega(t, \mathbf{x}^*[t], \mathbf{u}^*[t])\, dt +$$

$$\lim_{t\to\infty} V(t, \mathbf{x}^*[t]) - \lim_{t\to\infty} V(t, \mathbf{x}^\circ[t]) \leqslant \int_{t_0}^{\infty} \omega(t, \mathbf{x}^*[t], \mathbf{u}^*[t])\, dt$$

Q.E.D.

From the practical point of view the most interesting case is that, in which the limits appearing in (2.2) are equal to zero. Namely, from Theorem 1 follows

Corollary. Assume that the functions $\mathbf{u}^\circ(t, \mathbf{x}) \in K$ and $V(t, \mathbf{x})$ satisfying the conditions 1) — 3) of Theorem 1 exist and the following relation holds for any control $\mathbf{u}^*(t, \mathbf{x}) \in K$ satisfying the asymptotic $\mathbf{y}$-stability of the motion $\mathbf{x} = 0$:

$$\lim_{t\to\infty} V(t, \mathbf{x}^\circ[t]) = \lim_{t\to\infty} V(t, \mathbf{x}^*[t]) = 0 \tag{2.5}$$

Then the function $\mathbf{u}^\circ(t, \mathbf{x})$ solves the problem of optimal $\mathbf{y}$-stabilization in class K.

Notes. 1). Theorem 1 modifies the results of [1 — 3] in two aspects. Firstly the relation (2.2) is more general than the equality (2.5) the validity of which was guaranteed by the theorems of [1 — 3]. Secondly, in the theorems of [1 — 3] the asymptotic stability (with repsect to all or some of the variables) of the motion $\mathbf{x} = 0$ was established for $\mathbf{u} = \mathbf{u}^\circ(t, \mathbf{x})$ with help of the same function V which was used to establish the conditions 2) and 3) of Theorem 1 and the relation (2.5), although condition 1) of Theorem 1 can be verified using another Liapunov function (which may be a vector function) satisfying the conditions of any theorem of asymptotic $\mathbf{y}$-stability [4].

2). If $\lim V(t, \mathbf{x}^\circ[t]) = 0$ as $t \to \infty$, then by virtue of (2.2) Theorem 1 can be used only if the condition that $\lim V(t, \mathbf{x}^*[t]) \leqslant 0$ as $t \to \infty$, holds. When the function V is nonnegative, the latter inequality becomes an exact equality (see (2.5)).

3. Let us assume that the system (1.1) and the control quality criterion (1.3) are time independent and have respectively the following form:

$$\mathbf{x}^{\cdot} = \mathbf{X}(\mathbf{x}, \mathbf{u}) \tag{3.1}$$

$$J = \int_0^\infty \omega(\mathbf{x}[t], \mathbf{u}[t])\, dt \tag{3.2}$$

and continuous functions independent of t appear in the class $K = \{\mathbf{u}(\mathbf{x})\}$.

T h e o r e m 2. Assume that for any $\mathbf{u}(\mathbf{x}) \in K$ every solution of the system (3.1) originating in some neighborhood of the point $\mathbf{x} = 0$ is bounded, and let the functions $\mathbf{u}^\circ(\mathbf{x}) \in K$ and $V(\mathbf{x})$ be such that

1) $V(\mathbf{x}) \geqslant a(\|\mathbf{y}\|)$ where $a(r)$ is a continuous function monotonously increasing on $[0, H]$ and $a(0) = 0$;

2) $B[V, \mathbf{x}, \mathbf{u}^\circ(\mathbf{x})] = 0$ and

$$V^{\cdot}|_{\mathbf{u}=\mathbf{u}^\circ(\mathbf{x})} = -\omega(\mathbf{x}, \mathbf{u}^\circ(\mathbf{x})) \begin{cases} = 0 & \text{when } \mathbf{x} \in M \\ < 0 & \text{when } \mathbf{x} \overline{\in} M \end{cases}$$

3) $B[V, \mathbf{x}, \mathbf{u}(\mathbf{x})] \geqslant 0$ for any $\mathbf{u}(\mathbf{x}) \in K$;

4) the set [5] $M_0 = M \cap M_1$ does not contain any whole semi-trajectories ($t \in [0, \infty)$) of the system (3.1) when $\mathbf{u} = \mathbf{u}^\circ(\mathbf{x})$ where $M_1 = \{\mathbf{x}: V(\mathbf{x}) > 0\}$;

5) the relation $\lim V(\mathbf{x}^*[t]) = 0$ as $t \to \infty$ holds for any control $\mathbf{u}^*(\mathbf{x}) \in K$ ensuring the asymptotic $\mathbf{y}$-stability of the motion $\mathbf{x} = 0$.

Then the function $\mathbf{u} = \mathbf{u}^\circ(\mathbf{x})$ solves the problem of optimal $\mathbf{y}$-stabilization in class K.

P r o o f. By virtue of the conditions 1), 2) and 4) and Theorem 4 of [5], the motion $\mathbf{x} = 0$ of the system (3.1) with $\mathbf{u} = \mathbf{u}^\circ(\mathbf{x})$ is asymptotically $\mathbf{y}$-stable (and uniformly stable over $\{t_0, \mathbf{x}_0\}$), and $\lim V(\mathbf{x}^\circ[t]) = 0$ as $t \to \infty$. Subsequent application of the corollary of Theorem 1 completes the proof.

E x a m p l e. Let us consider a mechanical holonomic system with generalized coordinates $q_1, \ldots, q_n$ and time-independent constraints acted upon by the potential gyroscopic and certain other forces [3]

$$Q_i = \sum_{j=1}^{r} m_{ij}(\mathbf{q})\, u_j(\mathbf{q}, \mathbf{q}^{\cdot}) \tag{3.3}$$

$$u_j = 0 \text{ when } q_1 = \ldots = q_m = q_1^{\cdot} = \ldots = q_n^{\cdot} = 0 \; (m < n)$$

so that the equations of motion have the form

$$\frac{d}{dt}\frac{\partial T}{\partial q_i^{\cdot}} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + \sum_{j=1}^{n} g_{ij} q_j^{\cdot} + \sum_{j=1}^{r} m_{ij} u_j \quad (i = 1, \ldots, n;\; g_{ij} = -g_{ji}) \tag{3.4}$$

Using the total energy $H = T + U$ of the system as the Liapunov function, we obtain [3]

$$H^{\cdot} = \sum_{i=1}^{n} Q_i q_i^{\cdot} = \sum_{i=1}^{n} \sum_{j=1}^{r} m_{ij} u_j q_i^{\cdot} \tag{3.5}$$

Let us assume that [5 — 7]

1) when $\mathbf{u} = 0$, the system (3.4) admits a particular solution $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ (position of equilibrium);

2) the potential energy $U = U(q_1, \ldots, q_n)$ is positive-definite with respect to $q_1, \ldots, q_m$ $(m < n)$;

3) any mechanical considerations will show that the coordinates $q_{m+1}, \ldots, q_n$ are bounded in every perturbed motion (e.g. the coordinates may be angular (mod 2π) [7];

4) when $\mathbf{u} = 0$, the set $U(\mathbf{q}) > 0$ does not contain any positions of equilibrium of the system (3.4).

Following [3], we shall pose the problem of determining the controls $u_j = u_j^{\circ}$ ensuring the asymptotic stability of the position of equilibrium $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ with respect to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$ and minimizing the functional

$$J = \int_0^{\infty} \left(F(\mathbf{q}, \mathbf{q}^{\cdot}) + \sum_{i,j=1}^{r} \beta_{ij} u_i u_j \right) dt \tag{3.6}$$

in which $F(\mathbf{q}, \mathbf{q}^{\cdot})$ is a nonnegative function to be determined, and the quadratic form is a positive-definite function of the controls.

In [3] the conditions $B[H, \mathbf{q}, \mathbf{q}^{\cdot}, \mathbf{u}^{\circ}] = 0$ and $B[H, \mathbf{q}, \mathbf{q}^{\cdot}, \mathbf{u}] \geqslant 0$ were used to show that the optimal controls u_j° and the function F have the form

$$u_j^{\circ} = -\frac{1}{2} \sum_{k=1}^{r} \frac{\Delta_{kj}}{\Delta} \sum_{i=1}^{n} m_{ik} q_i^{\cdot} \tag{3.7}$$

$$F(\mathbf{q}, \mathbf{q}^{\cdot}) = \sum_{i,j=1}^{r} \beta_{ij} u_i^{\circ} u_j^{\circ} \tag{3.8}$$

Let us assume that the quadratic form (3.8) is positive-definite with respect to $q_1^{\cdot}, \ldots, q_n^{\cdot}$. Taking into account the fact that [3] $H^{\cdot} = -2F$ when $u_j = u_j^{\circ}$ we can conclude, using [5, 6], that the position of equilibrium $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ with $u_j = u_j^{\circ}$ is asymptotically stable with respect to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$ (and uniformly in $\{t_0, \mathbf{q}_0, \mathbf{q}_0^{\cdot}\}$), and $\lim H(q^{\circ}[t], q^{\cdot\circ}[t]) = 0$ as $t \to \infty$.

Let now u_j^* denote any control ensuring the asymptotic stability of the equilibrium $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ with respect to $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot}$. The set Γ^+ of the ω-limit points of any perturbed motion $\{\mathbf{q}^*[t], \mathbf{q}^{\cdot *}[t]\}$ is nonempty by virtue of condition 3), invariant [8] and consists therefore of the positions of equilibrium. Consequently, by virtue of 2) and 4) $U = 0$ on the set Γ^+ and this implies that $\lim H(\mathbf{q}^*[t], \mathbf{q}^{\cdot *}[t]) = 0$ as $t \to \infty$.

Using Theorem 2, we arrive at the following conclusion: the controls (3.7) solve the problem of optimal $(q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_n^{\cdot})$-stabilization of the position of equilibrium $\mathbf{q} = \mathbf{q}^{\cdot} = 0$ under the control quality criterion (3.6), (3.8).

4. When condition (2.2) ceases to hold, Theorem 1 becomes invalid and this can be confirmed in the following example. Let us consider a second order autonomous system (4.1) with the quality criterion (4.2).

$$y^{\cdot} = -y, \ z^{\cdot} = -zu^2 \tag{4.1}$$

$$J = \int_0^\infty (y^2 + z^2u^2)\,dt \tag{4.2}$$

We consider the positive-definite quadratic form $V = {}^1/_2\,(y^2 + z^2)$ as the optimal Liapunov function. Its derivative with respect to time is, by virtue of the system (4.1), $V^{\cdot} = -y^2 - z^2u^2$, therefore we have

$$B\,[V,\, y,\, z,\, u] \equiv 0 \tag{4.3}$$

Thus every control u satisfies the conditions 1) – 3) of Theorem 1. It follows therefore that the integral (4.2) must have the same value at all u. This is not however the case. When $u = u^{(1)} \equiv y$, the solutions of (4.1) have the form

$$y^{(1)}[t] = y_0e^{-t}, \quad z^{(1)}[t] = z_0 \exp\left[-\int_0^t y_0^2e^{-2\tau}d\tau\right]$$

from which on the basis of (4.3) we obtain

$$J\,|_{u=u^{(1)}} = \int_0^\infty y^2(1+z^2)\,dt = V(y_0,\ z_0) - \lim_{t\to\infty} V(y^{(1)}[t],\ z^{(1)}[t]) = \tag{4.4}$$

$$\frac{1}{2}\,[y_0^2 + z_0^2\,(1 - \exp\,(-y_0^2))]$$

When $u = u^{(2)} \equiv 0$, solutions of the system (4.1) have the form $y^{(2)}[t] = y_0e^{-t}$, $z^{(2)}[t] = z_0$, and from this we have, by virtue of (4.3),

$$J\,|_{u=u^{(2)}} = \int_0^\infty y^2\,dt = V(y_0, z_0) - \lim_{t\to\infty} V(y^{(2)}[t],\ z^{(2)}[t]) = \frac{1}{2}\,y_0^2 \tag{4.5}$$

Combining (4.4) and (4.5) we arrive at the inequality

$$J\,|_{u=u^{(1)}} > J\,|_{u=u^{(2)}} \text{ when } y_0 \neq 0,\ z_0 \neq 0 \qquad \text{Q. E. D.}$$

We take this opportunity to note that e.g. under the conditions of the Marachkov theorem [9] the function V need not tend to zero along the solutions. This can be illustrated by means of the following example. For the scalar equation $x^{\cdot} = -x$ the positive-definite function $V(t, x) = {}^1/_2\,(1 + \exp\,(2\,t))\,x^2$ which does not admit an infinitely small upper bound, has a negative-definite derivative $V^{\cdot} = -x^2$. Then along the solutions we have

$$\lim_{t\to\infty} V(t,\ x(t)) = \lim_{t\to\infty}\,[{}^1/_2\,(1 + e^{2t})\,x_0^2e^{-2t}] = {}^1/_2\,x_0^2 \neq 0 \quad \text{when} \quad x_0 \neq 0$$

Q. E. D.

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